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A dynamic model for a thermostat

X. ZOU[1], J. A. JORDAN[2] and M. SHILLOR[1]
[1]*Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, U.S.A.*
e-mail: shillor@oakland.edu
[2]*Henry Ford Health System, One Ford Place 3A, Detroit, Michigan 48202, U.S.A.*

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Abstract. A new mathematical model for the dynamic behavior of a thermostat situated in the cooling system of an engine is presented. The model is in the form of a system of delay-differential equations with hysteresis. An algorithm for numerical solutions of the model is described and some representative simulations are presented. The model predicts steady oscillatory solutions for certain ranges of the system parameters which are depicted for some parameter values. Also, it predicts initial overshooting of the engine coolant temperature. A Design of Experiments approach is used to characterize the regions in the model parameter space where oscillations occur and the results for those parameters which most influence the oscillations of solutions are presented.

Key words: thermostat, hysteresis, delay differential equations, car-engine cooling system, response surface.

1. Introduction

Thermostats in cars, technically known as ‘engine-coolant control valves’, are devices situated in the cooling system and control the engine’s operating temperature. They are set to adjust the cooling rate so that an essentially optimal acceptable operating temperature range is maintained in the engine block. Since the cooling system is designed to handle the extreme heat loads under which the engine can operate, only partial cooling capacity is required when operating under normal conditions. Indeed, if the cooling system were to operate at full capacity continuously, the engine would run too cold, well below its optimal operating temperature. The thermostat senses the coolant temperature and allows a larger or smaller flow of coolant through the radiator. In this way it is designed to keep the coolant temperature at almost constant value. When the engine and the coolant are cold, the thermostat is closed and the flow to the radiator is fully diverted to a bypass. Once the engine is running, the coolant temperature rises and the thermostat incrementally opens the flow path to the radiator. The flow splits between the part through the radiator and the part through the bypass. Under heavy load conditions, the thermostat opens completely and then the full cooling capacity of the radiator is realized.

In this paper we investigate the dynamic behavior of an automotive cooling system with a thermostat. Although thermostats are conceptually very simple, their dynamic behavior is not, since they exhibit *hysteresis*, *i.e.*, the way they open when the temperature rises differs from the way they close when the temperature decreases. Moreover, the cooling loop introduces delays into the system. Thus, interplay of hysteresis and delays is found, which is the novel feature addressed here. There exists considerable mathematical literature dealing with systems of ordinary differential equations with delays, the so-called delay differential equations, *cf*. [1]–[3] and reference therein. The topic of hysteresis has received considerable attention recently, *cf.* [4]–[6] and references therein. But, to our knowledge, the only papers where the two are combined are [7] and [8], where the mathematical analysis and numerical simulations are presented for simpler versions of the present model. We just mention that in [9] the problem discussed is with hysteresis, but without an explicit delay. In the engineering literature (see, *e.g.*, [10] or [11] and the references therein) the models are simpler. Often the dynamics of the thermostat alone is the main topic and not its interactions within the cooling loop.

The model that we present, which is of the ‘lumped mass’ type, attempts to capture this hysteretic behavior. It is derived from the balance of energy rates in each of the cooling system’s components. The model consists of a nonlinear system of three ordinary differential equations for the thermostat, the engine, and the radiator temperatures, together with a functional relation for the thermostat hysteresis opening function. It describes a thermostat situated at the engine outlet. A model for a thermostat situated at the engine inlet and a comparison of the behavior of the solutions to the two models will be investigated in the future.

We are interested in the behavior of the thermostat as an element of a system which is nonlinear and includes hysteresis and delays. Since our dynamical system is strongly nonlinear, a number of complex types of behavior are possible. Our main interest here is in identifying the parameter ranges where self-induced oscillations are found. From the practical point of view, that of the automotive industry, considerable oscillations in the engine cooling loop lead to deviations from the optimal engine operating conditions. These, in turn, may adversely affect fuel consumption, car emissions, passenger comfort and possibly accelerate engine wear. The long-term goal of our current research is optimal control of the engine operating temperature via control of the thermostat. Another aspect of practical importance is the temperature overshooting when starting from ambient temperature, which most solutions exhibit. The characterization of the various oscillations and investigation of possible chaotic behavior will be considered elsewhere.

The model and the algorithm can help the designer of a cooling system in simulating the behavior of the system before it is actually constructed.

The model is constructed in Section 2, where detailed energy-transfer rates are obtained in each of the system components and then assembled into a model. We fully explain the many simplifying assumptions which underlie the model and indicate how to take into account, in later stages, some features which we do not include in the present. In Section 3 we present the results of the analysis which has been performed in [7] characterizing fully the onset of oscillations in a simpler model. The numerical algorithm which has been used is described in Section 4, where typical solutions are depicted. Since the model contains many parameters and coefficients and does not have explicit or closed form solutions, we used the statistical method of Design of Experiments (DOE) to identify those parameters that influence oscillations most. The approach and the results can be found in Section 5. The graphical representation of the response surfaces gives a substantial insight into the regions in the parameter space where oscillations occur. A brief summary of our results is given in Section 6.

2. The model

In this section we present a dynamic model describing the time behavior of a thermostat which is situated in the engine’s cooling system. We consider the whole cooling loop which

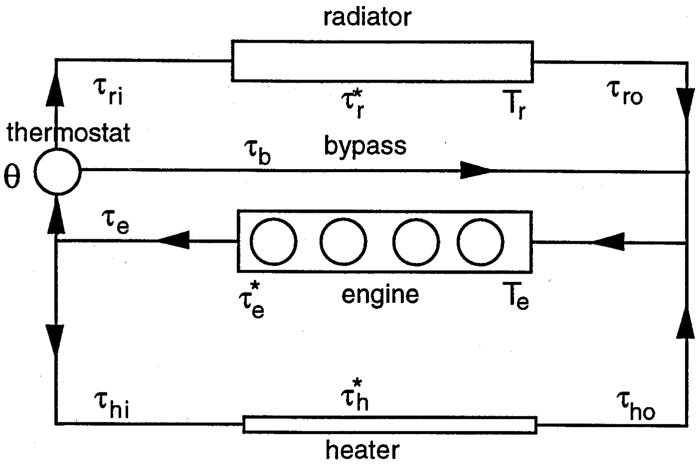


Figure 1. Schematic setting of the cooling system: engine, radiator, bypass, heater and thermostat.

contains the thermostat, since we are interested in the (nonlinear) interactions among the loop components.

A schematic diagram of the physical setting is depicted in Figure 1. Only the main elements in the cooling system are considered: radiator, engine, bypass and heater.

The basic assumptions underlying the model are that the energy source in the system is the engine; energy losses are in the radiator and heater; the internal thermostat temperature differs from the coolant temperature. We take into account the heat lost to exhaust gases and radiation only indirectly, as an estimated fraction of the engine power output. We describe the energy losses in the radiator and in the heater by Newton's law of cooling. The thermal interaction between the thermostat and the coolant is modeled similarly. The thermo-mechanical behavior of the thermostat is represented by the hysteresis graph, which we describe below. Finally, we do not take into account any mechanical or flow characteristics, except that the total flow is known and the flow at each loop component rearranges itself instantaneously with the thermostat opening.

We denote the coolant temperature at the thermostat by $T = T(t)$, as a function of time t (in sec), the temperature of the thermostat itself by $\theta(t)$, the radiator coolant temperature by $T_r = T_r(t)$ and the engine coolant temperature by $T_e = T_e(t)$ (in F^o).

We proceed to model the dynamic behavior of the system. The thermo-mechanical information characterizing the thermostat is given by its hysteresis graph β, depicted in Figure 2, which consists of the two curves f_R, f_L, and the *hysteresis region* $\mathcal{H}$ which lies between them. The hysteresis graph β describes the opening of the valve as a 'function' of the thermostat temperature θ. To be more precise, since β is a graph, we denote by $\omega = \omega(t)$ the fractional opening of the valve. When $\omega = 0$ it is closed, all the coolant flows via the bypass and heater loops, and there is no flow to the radiator. It is completely open when $\omega = 1$ leading to maximum flow and cooling by the radiator. The thermostat is partially open when $0 < \omega < 1$, and then only the fraction ω of the fluid flows to the radiator, while the rest flows via the bypass and heater.

The way hysteresis affects the dynamic behavior is as follows. The curve f_R describes the way the valve opens. When the state of the thermostat $(\theta(t), \omega(t))$ is on the curve f_R and the temperature is rising, then $\dot{\theta} > 0$, and the system will continue to move along the curve f_R. The valve closes along f_L. If the temperature is decreasing, *i.e.*, $\dot{\theta} < 0$, when $(\theta(t), \omega(t))$ is on

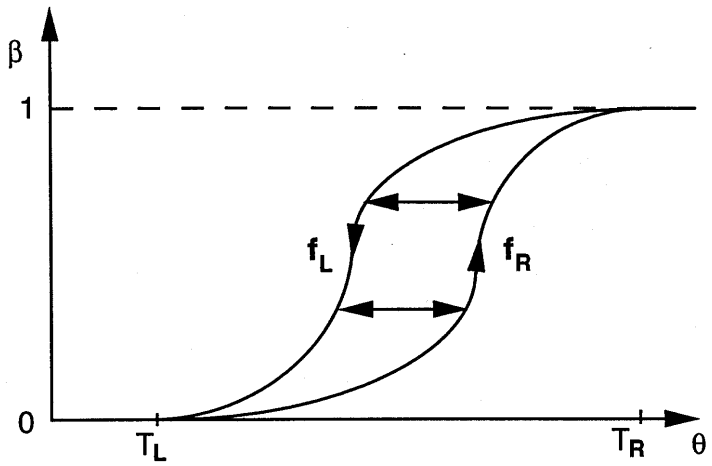


Figure 2. The hysteresis curves of β.

f_L, the thermostat will continue to move along the curve f_L. It may happen, however, that the system, while moving on f_R, reaches a time when $\dot{\theta}(t) = 0$ at a temperature $T_L < \theta(t) < T_R$, and afterwards the temperature decreases. To describe the system's behavior in such a case we assume the so-called 'generalized play model' (see, *e.g.*, [12] or [5]) by which the system moves on the horizontal segment connecting the curves f_R and f_L, while the valve opening is $\omega = \text{const.}$, until it reaches the curve f_L, on which it will continue moving down. We denote this behavior by the *hysteresis* operator H_β, and $\omega(t) = H_\beta(\theta(t))$.

We assume that $\mathcal{H}$ is filled with a family of horizontal segments connecting the two curves, the 'generalized play model'. Other choices of families of curves which fill the hysteresis region $\mathcal{H}$ lead to different models and possibly to different types of behavior. Eventually, the way $\mathcal{H}$ is filled has to be found experimentally.

Next, we give a short description of the thermostat. It is a device consisting of a metallic case enclosing a cavity full of wax in which a metallic pin is partially inserted. A spring has one end attached to the wax casing and the other end to the thermostat assembly. The exposed end of the pin is also attached to the superstructure. When the wax temperature reaches the value T_L (Figure 2), the wax starts melting which increases its volume and so it pushes the pin out against the restoring force of the spring. This motion of the pin opens the valve leading to coolant flow in the radiator loop. Once the wax is completely molten, at temperature T_R, the pin is displaced at its maximum and the valve is fully open. The energy needed to displace the pin and open the valve comes from the heat exchange between the coolant and the thermostat. This energy is used to raise the wax temperature to T_L, to melt it completely at T_R and then to raise the temperature of the melt.

The mathematical description of this process is as follows. Let c^s_{th} and c^l_{th} be the total heat capacities of the thermostat when fully closed and when fully open, respectively. These correspond to the cases when the wax is fully solid and fully molten. Since we deal with wax that starts softening at T_L and is fully molten at T_R we assume, for simplicity, that the heat capacity is a linear interpolation of the temperature in the range $[T_L, T_R]$, *i.e.*,

$$c_* = \begin{cases} c^s_{\text{th}} & \text{if } \theta \leqslant T_L, \\ c^s_{\text{th}} + (c^l_{\text{th}} - c^s_{\text{th}})\frac{\theta - T_L}{T_R - T_L} & \text{if } T_L \leqslant \theta \leqslant T_R, \\ c^l_{\text{th}} & \text{if } T_R \leqslant \theta. \end{cases} \tag{2.1}$$

This represents a continuous transition from solid to liquid when the wax is softening in the temperature range $[T_L.T_R]$.

Other choices are possible, for example, we may consider

$$c_* = c_*(t) = c^s_{\text{th}} + \omega(t)(c^l_{\text{th}} - c^s_{\text{th}}),$$

if we assume that ω also measures the fraction of molten wax. Here, the wax is assumed to exist as a mixture of solid and liquid particles with volume fractions $1-\omega$ and ω, respectively.

We now describe the different elements of the model. The setting and notation for the sources, temperatures, and delays are depicted in Figure 1.

Thermostat

We consider a small volume of coolant that arrives at the thermostat at time t from the engine at at temperature $T(t)$, while $\theta(t)$ is the thermostat internal temperature (that of the wax). Let h_{th} be the heat exchange coefficient between the coolant and the thermostat; the energy exchange rate is given by

$$\text{heat flux} = h_{\text{th}}(T(t) - \theta(t)).$$

This energy flux affects the thermostat temperature. It also affects the coolant temperature, but in real systems this effect is negligible, because the total thermal capacitance of the coolant is much greater than that of the thermostat.

Let λ denote the total latent heat of the wax in the thermostat, then $\lambda\omega(t)$ is the amount of energy needed to melt the fraction ω. Combining these leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_*\theta + \lambda\omega) = h_{\text{th}}(T - \theta).$$

Radiator

We consider a volume of coolant that arrives at the radiator from the thermostat at time t. Let T_r denote the (averaged) radiator temperature, T^r_{in} denote the coolant temperature at the inlet to the radiator and T^r_{out} the outlet temperature. The coolant flow time from the thermostat to the radiator is denoted by τ_{ri} and the time it takes to pass the radiator, the residence time, by τ^*_r. We assume that $T^r_{\text{in}}(t) = T(t - \tau_{ri})$, since there are no heat losses during the flow in the pipes between any of the system components. This assumption can be relaxed if the loss can be estimated. Let T_{amb} be the averaged ambient temperature passing through the radiator; using Newton's law of cooling we have

$$q_r = h_r(T_r(t) - T_{\text{amb}}),$$

where q_r is the flux of energy lost to the air and h_r is the radiator coefficient of heat exchange.

Let c be the heat capacity of a unit volume of coolant, v_r be the maximal flow rate through the radiator (when the thermostat is fully open), and V_r be the total volume of the coolant in the radiator, all three assumed constant. The energy rate balance in the radiator is: the energy influx is $cv_r\omega(t)T^r_{\text{in}}(t)$, the loss to the atmosphere is $h_r(T_r(t) - T_{\text{amb}})$ and the energy carried with the outflow is $cv_r\omega(t)T^r_{\text{out}}(t)$. We also assume that $T^r_{\text{out}}(t) = T_r(t)$ and that the outflow rate equals the inflow rate. Thus,

$$\begin{aligned} cV_r\frac{\mathrm{d}T_r(t)}{\mathrm{d}t} &= cv_r\omega(t)T^r_{\text{in}}(t) - cv_r\omega(t)T_r(t) - h_r(T_r(t) - T_{\text{amb}}) \\ &= cv_r\omega(t)(T(t - \tau_{ri}) - T_r(t)) - h_r(T_r(t) - T_{\text{amb}}). \end{aligned}$$

Bypass

We assume that the time delay in the bypass loop is the same as the total time delay in the radiator loop, accordingly

$$\tau_b = \tau_{ri} + \tau_r^* + \tau_{ro},$$

where τ_{ro} is the radiator to engine time of flow. The fluid inlet temperature is $T(t)$ and the outlet temperature (at the engine inlet) is $T(t - \tau_b)$. The energy flux at the outlet at time t is

$$\text{energy outflux} = cv_r(1 - \omega(t))T(t - \tau_b).$$

Heater

Consider a coolant volume that arrives at the heater from the thermostat. Let T_{in}^h denote the coolant temperature at the heater inlet and T_{out}^h the outlet temperature. Denote the flow delay from the thermostat to the heater by τ_{hi} and the residence time by τ_h^*. We assume that $T_{\text{in}}^h(t) = T(t - \tau_{hi})$, since there are no heat loses on the way. Let v_h be the flow rate in the heater loop, assumed to be constant, and let T_{ah} be the averaged ambient temperature at the heater (it may be different from the radiator ambient temperature). For the sake of simplicity we assume it is constant. Then,

$$\text{heat flux} = q_h = \alpha_s v_h h_h (T_{\text{in}}^h(t) - T_{\text{ah}}),$$

where h_h is the coefficient of heat exchange and we introduced the input coefficient α_s denoting the heater vent opening. When $\alpha_s = 1$ it is completely open, when $\alpha_s = 0$ the vent is completely closed and when $0 < \alpha_s < 1$, it is partially open. Then the heater energy rate balance is

$$\begin{aligned} cv_h T_{\text{out}}^h(t + \tau_h^*) &= cv_h T_{\text{in}}^h(t) - q_h \\ &= cv_h T_{\text{in}}^h(t) - \alpha_s v_h h_h (T_{\text{in}}^h(t) - T_{\text{ah}}) \\ &= (cv_h - \alpha_s v_h h_h) T_{\text{in}}^h(t) + \alpha_s v_h h_h T_{\text{ah}}. \end{aligned}$$

We set $\gamma = \alpha_s h_h / c$, and rewrite the equality as

$$T_{\text{out}}^h(t + \tau_h^*) = (1 - \gamma) T_{\text{in}}^h(t) + \gamma T_{\text{ah}}.$$

Engine

We denote the flow delay from the radiator to the engine by τ_{ro} and the delay from the heater to the engine by τ_{ho}. Then, at time t, the energy influx is from three sources: radiator, bypass and heater, given by

$$cv_r\omega(t)T_{\text{out}}^r(t - \tau_{ro}); \quad cv_r(1 - \omega(t))T(t - \tau_b); \quad cv_h T_{\text{out}}^h(t - \tau_{ho}),$$

respectively. To simplify the notation below let

$$\begin{aligned} \tau_r &= \tau_{ri} + \tau_r^* + \tau_{ro} + \tau_e^* + \tau_e \\ &= \tau_b + \tau_e^* + \tau_e \\ \tau_h &= \tau_{hi} + \tau_h^* + \tau_{ho} + \tau_e^* + \tau_e, \end{aligned}$$

where τ_e^* is the engine residence time, τ_e is the time delay between the engine and the thermostat and $\tau_b = \tau_{ri} + \tau_r^* + \tau_{ro}$. Then τ_r represents the total delay, or the time of round trip, for a coolant element from the thermostat via the radiator to the thermostat, while τ_h is the total delay in the heater loop.

The total energy influx to the engine is

$$\begin{aligned}\text{influx} \;=\; & c v_r \omega(t) T_{\text{out}}^r (t - \tau_{ro}) + c v_h T_{\text{out}}^h (t - \tau_{ho}) \\ & + c v_r (1 - \omega(t)) T(t - \tau_b).\end{aligned}$$

Now, we assume instantaneous mixing of the coolant coming from the radiator, bypass and heater, and we denote the entrance temperature as T_{in}^e, thus

$$\begin{aligned}c(v_r + v_h) T_{\text{in}}^e(t) \;=\; & c v_r \omega(t) T_{\text{out}}^r (t - \tau_{ro}) + c v_h T_{\text{out}}^h (t - \tau_{ho}) \\ & + c v_r (1 - \omega(t)) T(t - \tau_b).\end{aligned}$$

Using the expression for T_{out}^h and recalling that $T_{\text{out}}^r = T_r$ we obtain

$$\begin{aligned}c(v_r + v_h) T_{\text{in}}^e(t) \;=\; & c v_r \omega(t) T_r (t - \tau_{ro}) + c v_r (1 - \omega(t)) T(t - \tau_b) \\ & + c v_h [(1 - \gamma) T(t - \tau_h + \tau_e^* + \tau_e) + \gamma T_{\text{ah}}].\end{aligned}$$

Next, let V_e be the total volume of the coolant in the engine block, and let c_{bl} be the total heat capacity of the engine block (all that is not coolant), then

$$\begin{aligned}(c V_e + c_{bl}) \frac{\mathrm{d}T_e}{\mathrm{d}t} \;&=\; q_e - c(v_r + v_h) T_e(t) + c(v_r + v_h) T_{\text{in}}^e \\ &=\; q_e - c(v_r + v_h)(T_e(t) - T_{\text{in}}^e).\end{aligned}$$

Here q_e is the rate of engine heat rejection. Usually it is an estimated fraction of the nominal power rating of the engine.

We now return to the thermostat. Since we do not take into account the coolant heat loss in the thermostat, the inlet and outlet temperatures are the same, $T(t)$. Neglecting any heat losses on the way, we assume that

$$T(t) = T_{\text{out}}^e (t - \tau_e) = T_e(t - \tau_e).$$

Since the thermostat is physically situated inside the engine block, we assume that τ_e and τ_e^* are negligible and therefore $\tau_b = \tau_r$. Next, set

$$\tau = \max\{\tau_r, \tau_h\}.$$

Now, we collect and summarize our findings above in the following model.

Thermostat model
Find the functions $\{\theta,\ T_e,\ T_r,\ \omega\}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_* \theta + \lambda \omega) = h_{\text{th}}(T_e(t) - \theta(t)), \tag{2.2}$$

$$\omega(t) = H_\beta(\theta(t)), \tag{2.3}$$

$$(cV_e + c_{bl})\frac{\mathrm{d}T_e}{\mathrm{d}t} = q_e - c(v_r + v_h)(T_e(t) - T^e_{\mathrm{in}}(t)), \tag{2.4}$$

$$\begin{aligned} T^e_{\mathrm{in}}(t) = {} & (v_r\omega(t)T_r(t - \tau_{ro}) + v_r(1 - \omega(t))T_e(t - \tau_r) \\ & + v_h(1 - \gamma)T_e(t - \tau_h) + v_h\gamma T_{\mathrm{ah}})(v_r + v_h)^{-1}, \end{aligned} \tag{2.5}$$

$$cV_r\frac{\mathrm{d}T_r}{\mathrm{d}t} = cv_r\omega(t)(T_e(t - \tau_{ri}) - T_r(t)) - h_r(T_r(t) - T_{\mathrm{amb}}), \tag{2.6}$$

$$T_e(t) = T_{e0}(t), \qquad T_r(t) = T_{r0}(t), \quad -\tau \leqslant t \leqslant 0, \tag{2.7}$$

$$\theta(0) = \theta_0, \qquad \omega(0) = \omega_0. \tag{2.8}$$

Here, T_{e0}, T_{r0}, θ_0 and ω_0 are the initial conditions: T_{e0} and T_{r0} are given functions defined on the time interval $-\tau \leqslant t \leqslant 0$. This is due to the existence of the delays in the system. The initial condition for θ and ω is specified only at $t = 0$.

REMARK. If we wish to consider q_e, q_r, T_{amb} or T_{ah} as known functions varying with time we need to substitute $q_e = q_e(t)$, $q_r = q_r(t - \tau_r + \tau_{ri})$, $T_{\mathrm{amb}} = T_{\mathrm{amb}}(t)$ or $T_{\mathrm{ah}} = T_{\mathrm{ah}}(t - \tau_h)$ into (2.4)–(2.6).

The model consists of four dependent variables, one independent variable (time), four initial conditions, a hysteresis graph consisting of two functions, and 21 system parameters. It is possible to simplify it somewhat, but at the expense of the clear physical meaning of the parameters, so we present the nondimensional model in Appendix B.

3. Mathematical analysis of simplified models

The model (2.2)–(2.8) is rather complicated and its mathematical analysis is an open problem. Since it is strongly nonlinear, and includes hysteresis and delays, questions of existence, uniqueness and regularity of solutions need to be addressed. Also, it would be of interest to obtain sufficient conditions for the appearance of self-induced oscillations. In this section we present the analysis of two simplified versions of the model, where such sufficent conditions were obtained. The aim was to obtain insight into the appearance of intrinsic oscillations, and their relationship to the hysteres is curve and the system time delays. The full mathematical analysis can be found in [7]. Additional analysis and results are given in [8].

The first model may be thought of as describing a system with thermostat, engine, radiator and bypass, which has a (spatially uniform) temperature θ, thermostat opening ω, and a time delay τ. The model, set in nondimensional form, is: Find the pair $\{\theta, \omega\}$ such that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = q_e - q_r\omega(t - \tau), \tag{3.1}$$

$$\omega(t) = H_\beta(\theta(t)), \tag{3.2}$$

$$\theta(t) = \theta_0(t) \quad \text{for} \quad -\tau \leqslant t \leqslant 0, \tag{3.3}$$

$$\omega(-\tau) = \omega_0. \tag{3.4}$$

Here θ_0 is the initial temperature and ω_0 is the initial valve opening. Also, q_e is the engine thermal output and q_r is the cooling power of the radiator.

If we assume that θ_0 and the hysteresis curves are sufficiently smooth (say Lipschitz continuous) then the existence of a unique solution to (3.1)–(3.4) can be obtained by marching in time in steps of size τ. Moreover, the solution is smooth except possibly for a jump at $t = 0$.

We assume that $q_e/q_r < 1$. Then it is easy to see that any steady solution $\{\theta_*, \omega_*\}$ to (3.1)–(3.4) has to satisfy $\omega_* = q_e/q_r$, so that the right-hand side of (3.1) vanishes, and then $\theta_* \in [f_L^{-1}(\omega_*),\ f_r^{-1}(\omega_*)]$. Clearly, when $q_e/q_r > 1$ no steady state is possible.

We now describe the conditions for system oscillations. Let α be the minimum of the slopes of the hysteresis curves at the value q_e/q_r, *i.e.*,

$$\alpha = \min\left\{\frac{\mathrm{d}f_L}{\mathrm{d}\theta}(q_e/q_r),\quad \frac{\mathrm{d}f_R}{\mathrm{d}\theta}(q_e/q_r)\right\}.$$

Then, we have from [2, Theorem 3.2].

THEOREM 3.1. *Assume that*

$$q_r\,\tau\,\alpha > \frac{1}{e}. \tag{3.5}$$

Then the solution $\{\theta,\ \omega\}$ *of* (3.1)–(3.4) *that is not eventually constant, is oscillatory. In this case* θ *oscillates about the interval* $[f_L^{-1}(\omega_*),\ f_r^{-1}(\omega_*)]$, *and* ω *oscillates about* ω_*.

We remark that there may be oscillatory solutions for some values of $q_r\,\tau\alpha$ which do not satisfy (3.5). The proof in [7] is based on conditions for oscillations for delay differential equations.

Next, we present the model which is obtained from (3.1)–(3.4) when the cooling power of the radiator q_r, is not constant but chosen to satisfy Newton's cooling law

$$q_r = h(\theta - \theta_{\mathrm{amb}}),$$

where θ_{amb} is the air ambient temperature which is taken as zero, and h is the heat exchange coefficient.

The second model is: Find a pair $\{\theta,\ \omega\}$ such that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = q_e - h\omega(t-\tau)\theta(t-\tau), \tag{3.6}$$

$$\omega(t) = H_\beta(\theta(t)), \tag{3.7}$$

$$\theta(t) = \theta_0(t) \quad \text{for} \quad -\tau \leqslant t \leqslant 0, \tag{3.8}$$

$$\omega(-\tau) = \omega_0. \tag{3.9}$$

Let θ_L and θ_R be the solutions of

$$\frac{q_e}{h\theta} = f_L(\theta), \quad \text{and} \quad \frac{q_e}{h\theta} = f_R(\theta),$$

respectively. The 'singular values' of (3.6)–(3.9) are all θ in the interval $[\theta_L,\ \theta_R]$. Any steady solutions $\{\theta_*,\ \omega_*\}$ of the second model satisfy $\omega_* = q_e/(h\theta_*)$ and $\theta_* \in [\theta_L,\ \theta_R]$.

We have, see [2, Theorem 3.5]

THEOREM 3.2 ([CSSZ]). *A sufficient condition that all the solutions of* (3.6)–(3.9) *oscillate about some singular value is*

$$f_R(\theta_R) > \frac{1}{h\tau e}. \tag{3.10}$$

4. Numerical algorithm and simulations

In this section we present the algorithm we used to obtain numerical solutions for the model (2.2)–(2.8), and the representative results of our numerical experiments.

We employed the explicit Euler method to solve the system of ordinary differential equations. Although the algorithm was rather simple and more sophisticated methods of the predictor-corrector type could have been used, it performed well, exhibited numerical stability and ran quickly on a workstation. Thus, we felt that at this stage of the research further investment of time and effort to improve it was unnecessary.

Let Δt be the discretization time step. Let T_e^j, θ^j, T_r^j, and ω^j denote the values of these functions at time $t = j\Delta t$,

$$T_e^j = T_e(j\Delta t), \qquad \theta^j = \theta(j\Delta t), \qquad T_r^j = T_r(j\Delta t), \quad \text{and } \omega^j = \omega(j\Delta t).$$

These denote the finite differences approximations to the engine, thermostat, and radiator temperatures and thermostat opening at time $j\Delta t$, respectively. To take care of the system delays let d_{r2e}, d_{e2r}, d_h and d_b be the closest integers to $\tau_{ro}/\Delta t$, $\tau_{ri}/\Delta t$, $\tau_h/\Delta t$, and $\tau_r/\Delta t$, respectively. Also, let d be the closest integer to $\max\{\tau_r, \tau_h\}/\Delta t$.

The coolant volumes in various loops, heat rejection rates, coefficient values, values of delay times and flow rates are determined from the engine design specifications and experimentally determined curves, see *e.g.*, [13].

Since we deal with a system with delays, we initialize the variables by setting

$$T_e^j = T_{e0}(j\Delta t), \qquad T_r^j = T_{r0}(j\Delta t), \quad \text{for } -d \leqslant j \leqslant 0,$$

together with

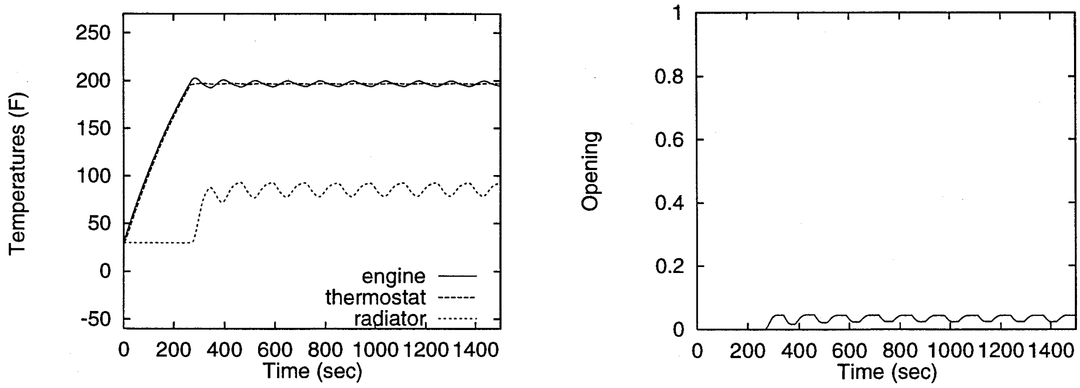
$$\theta^0 = \theta_0, \qquad \omega^0 = \omega_0.$$

Now, the algorithm proceeds in (time) steps. Assume that at the step j all the function values $\{\theta^k, T_e^k, T_r^k, \omega^k\}$ have been determined for $0 \leqslant k \leqslant j$. Then $\{\theta^{j+1}, T_e^{j+1}, T_r^{j+1}, \omega^{j+1}\}$ are determined as follows

First, we compute engine temperature from (2.4) and (2.5), using the formulas

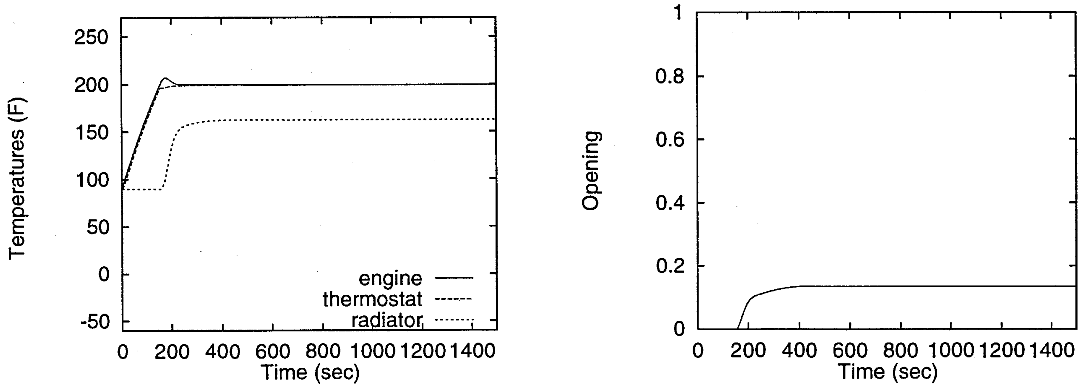
$$\begin{aligned} T_*^{j+1} &= q_e - c(v_r + v_h)T_e^j + cv_r w^j T_r^{j-d_{r2e}} + cv_r(1 - w^j)T_e^{j-d_b} \\ &\quad + cv_h[(1-\gamma)T_e^{t-d_h} + \gamma T_{\mathrm{ah}}], \end{aligned}$$

$$T_e^{j+1} = T_e^j + \frac{T_*^{j+1}\Delta t}{cV_e + c_{bl}}.$$



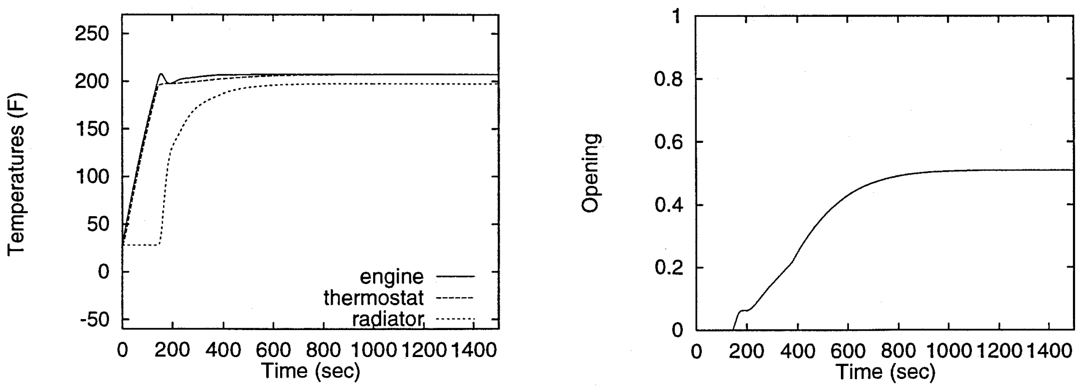
(a) (b)

Figure 3. $h_r = 162{\cdot}0$, $T_{\mathrm{amb}} = 30{\cdot}0$, rps $= 40{\cdot}0$ and $V = 9{\cdot}5$



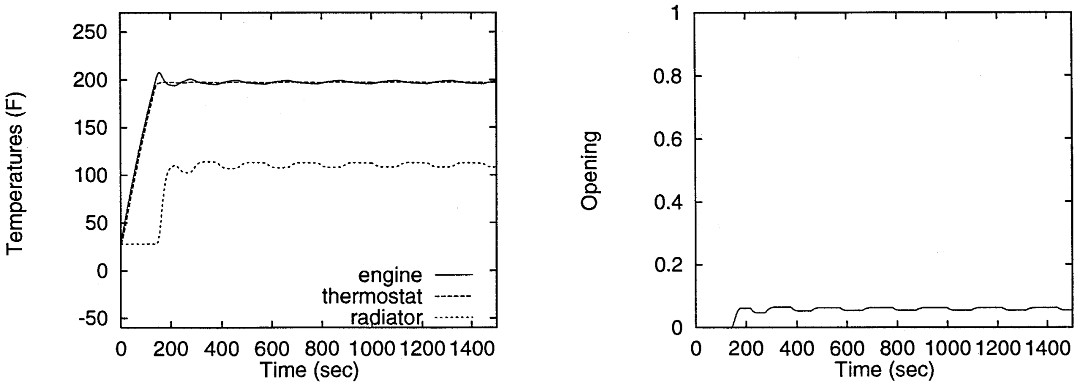
(a) (b)

Figure 4. $h_r = 162{\cdot}0$, $T_{\mathrm{amb}} = 90{\cdot}0$, rps $= 40{\cdot}0$ and $V = 9{\cdot}5$



(a) (b)

Figure 5. $h_r = 1112{\cdot}0$, $T_{\mathrm{amb}} = 28{\cdot}0$, rps $= 63{\cdot}0$ and $V = 9{\cdot}5$



(a) (b)

Figure 6. $h_r = 236{\cdot}0$, $T_{\text{amb}} = 28{\cdot}0$, rps $= 63{\cdot}0$ and $V = 9{\cdot}5$

Next, we compute the radiator temperature from (2.6) by

$$T_r^{j+1} = T_r^j + \Delta t(-h_r(T_r^j - T_{\text{amb}}) + cv_r w^j (T_e^{j-d_{e2r}} - T_r^j))(cV_r)^{-1}.$$

Then, θ^{j+1} is computed in a number of steps, since we have to take into account the temperature dependence of the heat capacity. Equation (2.2) is discretized as

$$c_*^{j+1}\theta^{j+1} = c_*^j\theta^j + \Delta t h_{\text{th}}(T_e^j - \theta^j) - \lambda(w^j - w^{j-1}).$$

Note that the derivative of ω was shifted one time-step backward to keep the scheme explicit. To approximate c_*^{j+1} we use θ^j, thus

if $\theta^j \geqslant T_R$ then $c_*^{j+1} = c_l$;
if $\theta^j \leqslant T_L$ then $c_*^{j+1} = c_s$;
if $T_L \leqslant \theta^j \leqslant T_R$ then

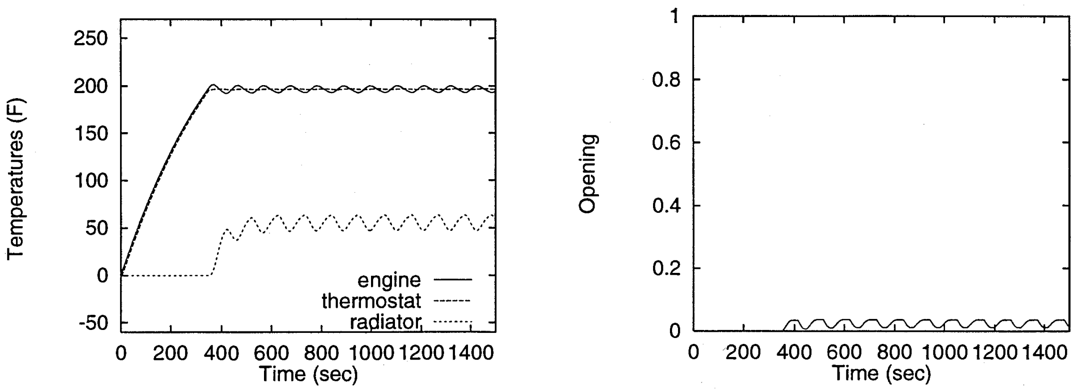
$$c_*^{j+1} = (c_s T_R - c_l T_L + 2(c_l - c_s)\theta^j)(T_R - T_L)^{-1}.$$

Finally, we compute the new opening ω^{j+1} from the hysteresis curves and the play model assumption as follows: if $\omega^j = f_R(\theta^j)$ and ω is increasing, then the opening follows f_R, when $\omega^j = f_L(\theta^j)$ and θ is decreasing the opening follows the curve f_L, otherwise the opening is constant, at the same value as in the previous time step. In terms of coding we use

if $(\omega^j = f_R(\theta^j))$ then
  if $(\theta^{j+1} > \theta^j)$ then
    $\omega^{j+1} = f_R(\theta^{j+1})$
  else
    $\omega^{j+1} = \omega^j$
  endif
else if $(\omega^j = f_L(\theta^j))$ then
  if $(\theta^{j+1} < \theta^j)$ then

Table 1. The values of the simulation constants.

Notation	Value	Unit	Explanation
c_s	300·0	(Watt)(Sec)/F	solid wax heat capacity
c_l	1200·0	(Watt)(Sec)/F	liquid wax heat capacity
λ	200·0	(Watt)(Sec)	latent heat of wax
h_{th}	50·0	(Watt)/F	heat exchange coefficient of thermostat
h_r	174·0	(Watt)/F	heat exchange coefficient of radiator
c	1800·0	(Watt)(Sec)/(Lit)/F	heat capacity of unit coolant
c_{bl}	9900·0	(Watt)(Sec)/F	engine block heat capacity
ω_0	0·0		initial thermostat opening



(a) (b)

Figure 7. $h_r = 174{\cdot}0$, $T_{\mathrm{amb}} = 0{\cdot}0$, rps $= 46{\cdot}0$ and $V = 14{\cdot}5$

$\quad\omega^{j+1} = f_L(\theta^{j+1})$

else

$\quad\omega^{j+1} = \omega^j$

endif

else if $(f_L(\theta^j) > \omega^j > f_R(\theta^j))$ then

$\quad\omega^{j+1} = \omega^j$

endif

if $(\omega^{j+1} < f_R(\theta^{j+1}))$ set $\omega^{j+1} = f_R(\theta^{j+1})$

if $(\omega^{j+1} > f_L(\theta^{j+1}))$ set $\omega^{j+1} = f_L(\theta^{j+1})$.

We now describe a number of typical or interesting simulations. Table 1 lists all the constant values used in this experiment. In Figures 3–8 we depict our simulations of the model. In each the engine rps (revolutions per second) is constant, and the numerical values of some of the parameters are indicated below each figure. The engine, radiator and thermostat temperatures are shown on the left, the thermostat opening on the right.

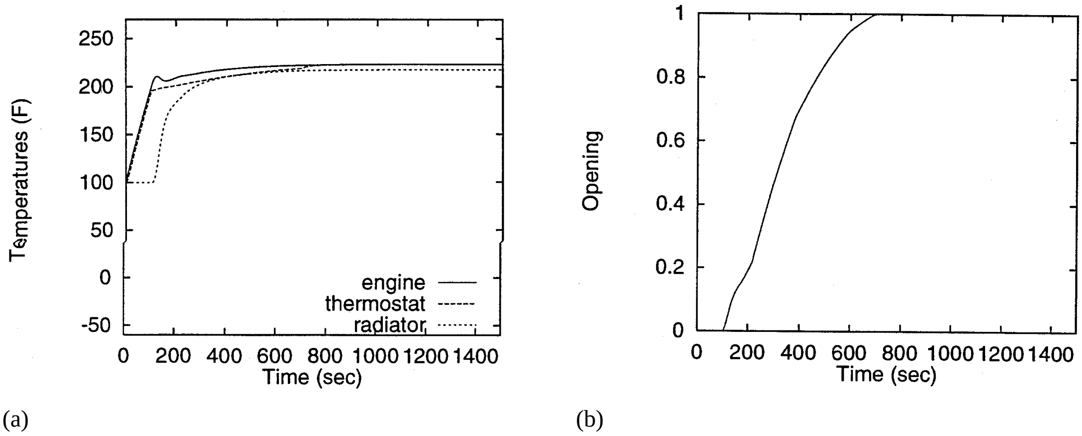


Figure 8. $h_r = 174{\cdot}0$, $T_{\text{amb}} = 100{\cdot}0$, rps $= 60{\cdot}0$ and $V = 14{\cdot}5$

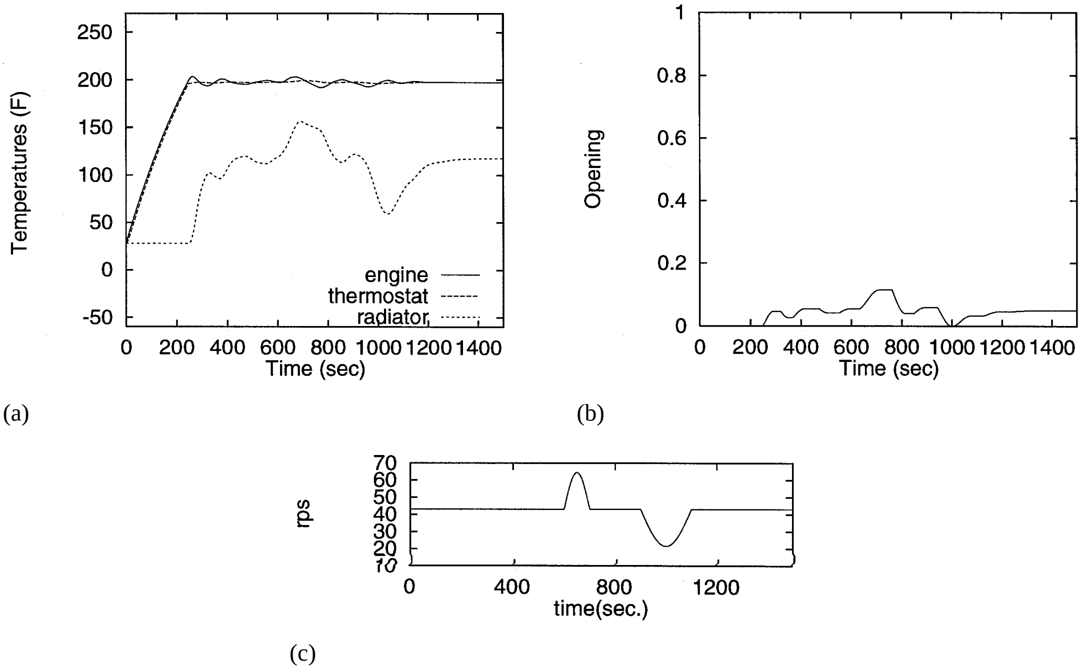


Figure 9. The case with variable rps; $h_r = 112{\cdot}0$, $T_{\text{amb}} = 28{\cdot}0$, rps $= 40{\cdot}0$ and $V = 9{\cdot}5$

In Figure 3 system oscillations with a period of 1·5 minutes can be seen. The initial temperature overshooting is small. In Figures 4 and 5 there are no oscillations. Steady opening and temperatures are achieved within about 5 minutes from start. Oscillations which clearly decay with time are depicted in Figure 6. In Figure 7 the ambient temperature is very low, leading to oscillations. The load and rps in Figure 8 are high so the thermostat opens fully. Finally, in Figure 9 we depict the system response to variable rps. The rps *vs.* time curve is shown on the bottom, and it includes one sharp increase and one sharp decrease. The system response is seen to be rather complicated. The figure represents more realistic ‘in town’ driving conditions.

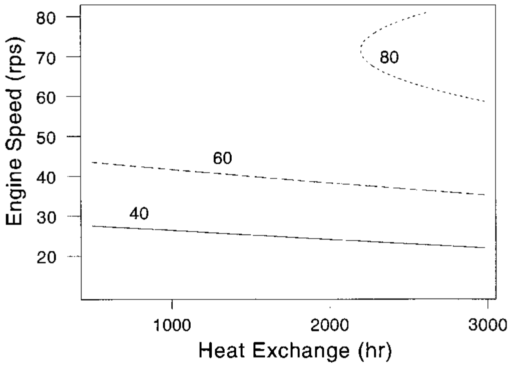


Figure 10. The contour plot of number of oscillations in the h_r-rps plane, with $V = 14{\cdot}0$ and $T_{\text{amb}} = 56.0$.

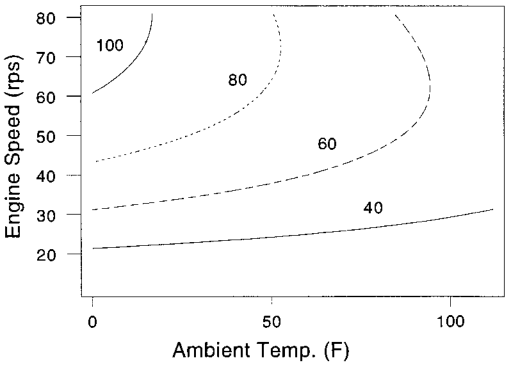


Figure 11. The plot in the T_{amb}-rps plane, with $V = 14{\cdot}0$ and $h_r = 174{\cdot}0$.

A typical run of 1500 seconds of simulations took about 2 seconds on a DEC ALPHA workstation (model 2100).

5. Design of experiments

The problem (2.2)–(2.8) does not have closed-form solutions and therefore we do not have an explicit way to determine the dependence of the solutions on system data. In view of the large number of coefficients or parameters in the problem, we decided to perform a DOE analysis to identify those parameters which have major influence on system oscillations and temperature overshooting. For general reference on DOE, see, *e.g.*, [14]. We present the results of our investigation of the effects of four main parameters, factors in statistical language, on four responses of the model. The parameters chosen for the study were the temperature of the outside air T_{amb}, total coolant volume V, radiator heat exchange coefficient h_r, and engine speed rps (revolutions per second). The coolant flow rates, delays and engine power generation were chosen to depend on rps. The main responses of interest were the number of oscillations of the engine coolant temperature, peak coolant temperature, difference between peak temperature and minimum temperature after the first peak temperature, and range in temperature after the first peak temperature. The first temperature was an overshooting, much bigger than any other peak, therefore we choose to discount it. The construction of a measure of temperature oscillations was accomplished by the generating function

$$\sum_n \{\text{if}(\text{sign}\,(T^n - T^{n-10})) = \text{sign}\,(T^{n-1} - T^{n-11}) \text{ then } 0 \text{ else } 1)\}.$$

That is, we compare the temperature value at time $t_n = n\Delta t$ with the value ten time steps earlier at time $t_{n-10} = (n - 10)\Delta t$. If the sign of the difference is different from the sign of the difference at the previous time step, we count it as an oscillation, since there is a change in the trend. The choice of ten time steps was found to give satisfactory results.

The values for each of the parameters were chosen to cover the entire range of realistic operating conditions. The response function of the system with respect to the chosen parameters was assumed to be nonlinear and included interactions among the factors.

All the computer simulations were run at a constant rps and started from the ambient temperature.

We employed a 4 factor central composite design with a single center point for the numerical experiments. Replication of a center point and randomization of run order were unnecessary, since the experiments consisted of computer runs with no test-retest error. Moreover, there was no need to attempt to reduce the size of the experiments, since a large number of experimental runs could be accomplished in a short time with little expense. A single run consisted of simulating 1500 seconds of system operation under the experimental conditions. The results of fitting a quadratic response model to the number of oscillations $\bar{y}$ and examination of residual plots suggested that the following model was adequate

$$\begin{aligned}\bar{y} \;=\;& \underset{(13{\cdot}05)}{1{\cdot}608} - \underset{(0{\cdot}2856)}{1{\cdot}657V} + \underset{(0{\cdot}0021)}{0{\cdot}004h_r} + \underset{(0{\cdot}1537)}{0{\cdot}080T_{\text{amb}}} + \underset{(0{\cdot}4230)}{2{\cdot}950\,\text{rps}}\\ & \underset{(0{\cdot}0041)}{-0{\cdot}017(\text{rps})^2} - \underset{(0{\cdot}0032)}{0{\cdot}008(T_{\text{amb}} \times \text{rps})}.\end{aligned} \tag{5.1}$$

The standard error of each coefficient is given in parentheses under the coefficient.

This model was chosen since the coefficients for the other interactions and quadratic terms were found to be not significantly different from zero, which means these terms could be replaced by zero. These results seem to agree with the current understanding of cooling systems behavior. The availability of statistical models, such as the one above, enabled a rapid and efficient investigation of the entire range of operating conditions. Figure 10 shows the contour plot of the number of oscillations as a function of rps and heat exchange coefficient h_r. This plot, as well as Figure 11, show a relationship that agrees with the experience of engineers in the field. Excessive cooling capacity leads to increased propensity for system oscillations. We note that increasing the heat exchange coefficient or decreasing the ambient temperature result in increase in the system cooling power.

The second phase of the DOE was designed to investigate the consistency of the results when a smaller region in the parameter space is studied. When the response surface has a complex structure, with many local maxima and minima, the model for the smaller region could contain many features not encompassed by the larger model. The experiment focused on the region with the largest number of oscillations. An orthogonal central composite design with all 2^4 factorial points, eight axial points and a single center point, was employed to investigate the oscillations of the thermostat temperature.

The experimental approach was the same as in the first model. The response y_i of number of oscillations was recorded for each experimental run. The simplified model was chosen since the coefficients for the other interactions and the quadratic terms were found to be not significantly different from zero. The model over the subregion with many oscillations reduced to

$$\begin{aligned}\bar{y} \;=\;& \underset{(8{\cdot}08)}{-27{\cdot}79} - \underset{(0{\cdot}081)}{3{\cdot}75V} + \underset{(0{\cdot}004)}{0{\cdot}0022h_r} + \underset{(0{\cdot}111)}{0{\cdot}0085T_{\text{amb}}} + \underset{(0{\cdot}239)}{2{\cdot}47\,\text{rps}}\\ & \underset{(0{\cdot}0018)}{-0{\cdot}0107(\text{rps})^2} - \underset{(0{\cdot}0017)}{0{\cdot}0058(T_{\text{amb}} \times \text{rps})}.\end{aligned} \tag{5.2}$$

The model (5.2), for the reduced region, appears to be very different from the model (5.1) for the entire region. However, the fitted values in the region of study are similar to the larger

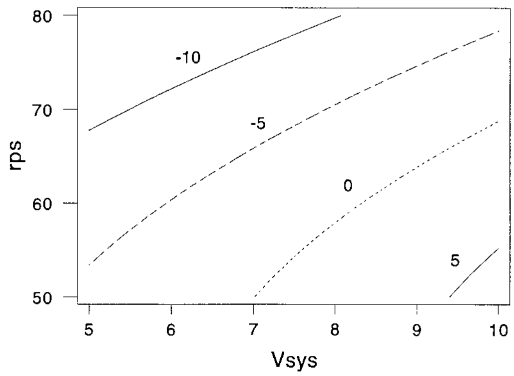


Figure 12. The reduced region plot in the V-rps plane with $h_r = 3000{\cdot}0$ and $T_{\mathrm{amb}} = 10{\cdot}0$.

model where the regions overlap. That is, the predictions of both models when restricted to the subregion of (5.2) agree well. The variance between the models increases as a function of distance from the center point of (5.2). No evidence of local maxima or minima was discovered in the subregion. The difference between the model for the reduced region and the model for the entire region was compared over the reduced region. Figure 12 is a contour plot of the difference between the larger model and the model for the reduced region.

6. Conclusions

We have developed a model for the dynamical behavior of automotive engine outlet thermostats. The model is set as a system of three delay-differential equations for component temperatures and a functional relationship for the thermostat opening. The latter represents the hysteresis thermo-mechanical behavior of the valve. The model is based on energy flow rates. A computer code, based on explicit time marching, has been employed to generate numerical solutions to simulate the model behavior. The numerical algorithm was rather unsophisticated, but served our purposes well. We found that the model predictions agree qualitatively with what cooling systems engineers have been finding experimentally. The model predicts the appearance of intrinsic oscillations in certain regions in the parameter space. Some of these oscillations decay with time, others seem to go on undiminished. Moreover, when the starting point is the ambient temperature the model predicts initial temperature overshoot. To investigate the dependence of the oscillations on the various model parameters we employed DOE and obtained a satisfactory characterization of the parameter regions where oscillations occur.

We conclude that the model may be used as a tool helping the automotive engine and cooling system designer to better characterize the systems vulnerability to oscillations and to find ways to minimize them. Moreover, many additions and modifications are possible to include other effects, to add additional interactions and to include other system components. However, these come at the expense of making the model cumbersome and harder to analyze. Finally, since the fluid flow was not considered, it may be of interest to add a stochastic component to the model representing turbulent flow in parts of the system.

The model is complex and a mathematical analysis is needed to fully understand its properties. The novelty here is the combination of delay-differential equations with hysteresis. Clearly, the nature of the oscillations needs further investigation. Indeed, we do not know how to distinguish between oscillations that decay with time, those that are periodic, quasiperiodic and those that are chaotic.

The model seems to be interesting, useful and in need of further investigation.

Appendices. A. The list of coefficients

We have collected all the model parameters here.

c heat capacity of one liter of coolant $\left[\dfrac{\text{watt}\cdot\text{sec}}{\text{lit}\cdot\text{F}}\right]$,

c_{bl} total heat capacity of the engine block [watt · sec/F],

c^l_{th} total heat capacity of the liquid wax [watt · sec/F],

c^s_{th} total heat capacity of the solid wax [watt · sec/F],

h_r heat exchange coefficient of the radiator [watt/F],

h_{th} heat exchange coefficient of the thermostat [watt/F],

q_e engine heat rejection [watt],

T_{amb} ambient (air) temperature [F],

T_{ah} passenger compartment air temperature [F],

T_L thermostat opening temperature [F],

T_R thermostat temperature when fully open [F],

v_r coolant flow rate in the radiator loop [lit/sec],

v_h coolant flow rate in the heater loop [lit/sec],

V total coolant volume in the system [lit],

V_e total coolant volume in the engine [lit],

V_r total coolant volume in the radiator [lit],

α_s fractional heater compartment opening,

γ $= \alpha_s h_h/c$,

λ total latent heat of the wax [watt · sec],

τ_h round trip time (delay) in the heater loop [sec],

τ_r round trip time (delay) in the radiator loop [sec],

τ_{ri} trip time (delay) from thermostat to radiator [sec]

τ_{ro} trip time (delay) from radiator to engine [sec].

B. The model in nondimensional form

We present the model in a nondimensional form, to indicate the combinations of coefficients that are controlling the problem. So we scale the variables in the model (2.2)–(2.8). We denote the variables in (2.2)–(2.8) with a tilde, and the new ones without.

We start with the scaling of the time and the temperatures as follows

$$t = \frac{h_{\text{th}}}{(c^l_{\text{th}} - c^s_{\text{th}})}\tilde{t} = A\tilde{t}, \qquad \theta = \frac{\tilde{\theta} - T_L}{T_L}, \qquad T_e = \frac{\tilde{T}_e - T_L}{T_L}, \qquad Tr = \frac{\tilde{T}_r - T_L}{T_L}.$$

T_L is the temperature at which the thermostat begins to open. We could as well scale the temperatures with T_{amb}. Also, note that A scales the time. Next, we introduce the new constants

$$a^* = \frac{c_*}{(c^l_{\text{th}} - c^s_{\text{th}})}, \tag{B.1}$$

$$b^* = \frac{\lambda}{T_L(c^l_{\text{th}} - c^s_{\text{th}})}, \tag{B.2}$$

$$d_e^* = \left(V_e + \frac{c_{bl}}{c}\right)\frac{h_{\text{th}}}{v_r(c_{\text{th}}^l - c_{\text{th}}^s)}, \tag{B.3}$$

$$d_r^* = \frac{V_r h_{\text{th}}}{v_r(c_{\text{th}}^l - c_{\text{th}}^s)}, \tag{B.4}$$

$$e^* = \frac{\gamma v_h}{v_r}\left(1 - \frac{T_{\text{ah}}}{T_L}\right), \tag{B.5}$$

$$h_r^* = \frac{h_r}{c v_r}, \tag{B.6}$$

$$T_a = 1 - \frac{T_{\text{amb}}}{T_L}, \qquad v^* = \frac{v_h}{v_r}, \tag{B.7}$$

$$q_e^* = \frac{q_e}{c T_L v_r}, \tag{B.8}$$

$$\tilde{\tau} = \frac{\tau_r}{A}, \qquad \tilde{\tau}_{ri} = \frac{\tau_{ri}}{A}, \qquad \tilde{\tau_{ro}} = \frac{\tau_{ro}}{A}, \qquad \tilde{\tau}_h = \frac{\tau_h}{A}, \tag{B.9}$$

where

$$c_* = \begin{cases} c_{\text{th}}^s & \text{if } \theta \leqslant 0, \\ \frac{1}{2}(c_{\text{th}}^s + c_{\text{th}}^1) & 0 \leqslant \theta \leqslant \frac{T_R - T_L}{T_L}, \\ c_{\text{th}}^l & \text{if } \frac{T_R - T_L}{T_L} \leqslant \theta. \end{cases} \tag{B.10}$$

The hysteresis graph β is scaled as

$$\beta_* = \beta_*(\theta(t), \dot{\theta}(t)) = \beta(T_L(\theta(t) + 1), \dot{\theta}(t)). \tag{B.11}$$

Now we can present the model (2.2)–(2.8) in a dimensionless form.

Thermostat model

We seek the functions $\{\theta, T_e, T_r, \omega\}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a^*\theta + b^*\omega) = T_e(t) - \theta(t), \tag{B.12}$$

$$\omega(t) = H_{\beta_*}(\theta(t)), \tag{B.13}$$

$$\mathrm{d}_e^* \frac{\mathrm{d}T_e}{\mathrm{d}t} = q_e^* - (1 + v^*)T_e(t) + \omega(t)T_r(t - \tau_{ro}) \tag{B.14}$$

$$+ (1 - \omega(t))T_e(t - \tau_r) \tag{B.15}$$

$$+ v^*(1 - \gamma)T_e(t - \tau_h) - e^*, \tag{B.16}$$

$$\mathrm{d}_r^* \frac{\mathrm{d}T_r}{\mathrm{d}t} = \omega(t)(T_e(t - \tau_{ri}) - T_r(t)) - h_r^*(T_r(t) + T_a), \tag{B.17}$$

$$T_e(t) = T_{e0}(t), \qquad T_r(t) = T_{r0}(t), \quad -\tau \leqslant t \leqslant 0, \tag{B.18}$$

$$\theta(0) = \theta_0, \qquad \omega(0) = \omega_0. \tag{B.19}$$

Here the initial conditions are scaled appropriately.

The nondimensional model has only 15 parameters, but this reduction seems to make little difference, and it is not clear if there are better choices of scaling.

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